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Correction of order three for the expansion of two dimensional electromagnetic fields perturbed by the presence of inhomogeneities of small diameter

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Abstract

The derivation of the correction of order 3 for the expansion of 2 dimensional electromagnetic fields perturbed by the presence of dielectric inhomogeneities of small diameter was completed in [3]. However previous numerical work such as that in [6] and in [14] do not corroborate the existence of these correcting terms. The inhomogeneities used in all those numerical simulations were collections of ellipses. In this paper we propose to elucidate this discrepancy. We prove that the correction of order 3 is zero for any inhomogeneity that has a center of symmetry. We present numerical experiments for asymmetric inhomogeneities. They illustrate the importance of the correction of order 3. Finally we prove that numerical schemes based on the usual quadrature for solving mixed linear integral equations on a smooth contour with smooth integration kernels and kernels involving logarithmic singularities preserve at the discrete level the fact that correcting terms of order 3 are zero for inhomogeneities that are symmetric about their center.

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1. Introduction

Let Ω be a smooth domain in \mathbb{R}^2 . The outward unit normal to $\partial\Omega$ is denoted by ν . Assume that Ω contains a finite number of inhomogeneities, each of the form $z_j + \alpha B_j$, where $B_j \subset \mathbb{R}^2$ is a bounded, smooth domain containing the origin. The total collection of inhomogeneities is

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$$\mathcal{B}_\alpha = \cup_{j=1}^m \{z_j + \alpha B_j\}.$$

The points $z_j \in \Omega, j = 1, \dots, m$, which determine the location of the inhomogeneities, are assumed to satisfy the following inequalities:

$$|z_j - z_l| \geq c_0 > 0 \quad \forall j \neq l \quad \text{and} \quad \text{dist}(z_j, \partial\Omega) \geq c_0 > 0 \quad \forall j. \tag{1}$$

Assume that $\alpha > 0$, the common order of magnitude of the diameters of the inhomogeneities, is sufficiently small, that these inhomogeneities are disjoint, and that their distance to $\mathbb{R}^2 \setminus \overline{\Omega}$ is larger than $c_0/2$. Let μ_0 and ε_0 denote the permeability and the permittivity of the background medium. Let them satisfy the usual requirements $\mu_0 > 0$ and $\text{Re } \varepsilon_0 > 0, \text{Im } \varepsilon_0 \geq 0$. Let $\mu_j > 0$ and $\text{Re } \varepsilon_j > 0, \text{Im } \varepsilon_j \geq 0$ denote the permeability and the permittivity of the j th inhomogeneity, $z_j + \alpha B_j$. Introduce the piecewise constant magnetic permeability

$$\mu_\alpha(x) = \begin{cases} \mu_0, & x \in \Omega \setminus \overline{\mathcal{B}_\alpha}, \\ \mu_j, & x \in z_j + \alpha B_j, \quad j = 1, \dots, m. \end{cases} \tag{2}$$

If we allow the degenerate case $\alpha = 0$, then the function $\mu_0(x)$ equals the constant μ_0 . The piecewise constant electric permittivity, $\varepsilon_\alpha(x)$ is defined analogly. Consider the solutions to the time-harmonic Maxwell’s equations with TE symmetry and $\exp(-i\omega t)$ time dependence. Let E_α be the electric field (or rather, the transverse strength) in the presence of the inhomogeneities. It satisfies the Helmholtz equation

$$\text{div} \left(\frac{1}{\mu_\alpha} \text{grad } E_\alpha \right) + \omega^2 \varepsilon_\alpha E_\alpha = 0 \text{ in } \Omega, \tag{3}$$

with the boundary condition $E_\alpha = f$ on $\partial\Omega$, where $\omega > 0$ is a given frequency. Eq. (3) contains implicitly the conditions on E_α across the boundary ∂B_j . E_α is required to be continuous across ∂B_j and its normal derivative has to satisfy the following jump condition:

$$\frac{1}{\mu_0} \frac{\partial E_\alpha}{\partial \nu} \Big|_+ - \frac{1}{\mu_j} \frac{\partial E_\alpha}{\partial \nu} \Big|_- = 0.$$

The electric field, E_0 , in the absence of any inhomogeneities, satisfies the following equation:

$$\Delta E_0 + k^2 E_0 = 0 \text{ in } \Omega, \tag{4}$$

where $k^2 = \omega^2 \mu_0 \varepsilon_0$, with $E_0 = f \in H^{\frac{1}{2}}(\partial\Omega)$ on $\partial\Omega$. In order to insure well-posedness we shall assume that

$$\begin{aligned} &k^2 \text{ is not an eigenvalue for the operator } -\Delta \text{ in } L^2(\Omega) \\ &\text{with homogeneous Dirichlet boundary conditions.} \end{aligned} \tag{5}$$

Under this assumption it was proved in [13] that Eq. (3) together with the boundary condition $E_\alpha = f$ on $\partial\Omega$ is a well-posed problem for α small enough.

Let $G_D(x, y)$ be the Dirichlet Green function in Ω corresponding to a Dirac mass at the point x . That is, G_D is the solution to

$$\begin{cases} -(A_y + k^2)G_D(x, y) = \delta_x \text{ in } \Omega, \\ G_D = 0 \text{ on } \partial\Omega. \end{cases} \tag{6}$$

It is possible to show that $G_D(x, y) = G_D(y, x)$ for x and y in Ω such that $x \neq y$.

The electric field E_α satisfies the integral representation formula

$$E_\alpha(x) = E_0(x) + \sum_{j=1}^m \left(1 - \frac{\mu_0}{\mu_j}\right) \int_{z_j+\alpha B_j} \text{grad} E_\alpha(y) \cdot \text{grad}_y G_D(x, y) \, dy + k^2 \sum_{j=1}^m \left(\frac{\varepsilon_j}{\varepsilon_0} - 1\right) \int_{z_j+\alpha B_j} E_\alpha(y) G_D(x, y) \, dy. \tag{7}$$

However the following representation formula is also sometimes convenient to use:

$$E_\alpha(x) - E_0(x) - \int_{\partial\Omega} G^k(x, y) \left(\frac{\partial E_\alpha(y)}{\partial v(y)} - \frac{\partial E_0(y)}{\partial v(y)}\right) \, ds(y) = \sum_{j=1}^m \left(1 - \frac{\mu_0}{\mu_j}\right) \int_{z_j+\alpha B_j} \text{grad} E_\alpha(y) \cdot \text{grad}_y G^k(x, y) \, dy + k^2 \sum_{j=1}^m \left(\frac{\varepsilon_j}{\varepsilon_0} - 1\right) \int_{z_j+\alpha B_j} E_\alpha(y) G^k(x, y) \, dy, \tag{8}$$

for x in Ω , where the free space Green’s function

$$G^k(x, y) = \frac{i}{4} H_0^1(k|x - y|). \tag{9}$$

We then take the limit of the normal derivative of the above expression as x approaches $\partial\Omega$ to find

$$\frac{1}{2} \left(\frac{\partial E_\alpha(x)}{\partial v(x)} - \frac{\partial E_0(x)}{\partial v(x)}\right) - \int_{\partial\Omega} \frac{\partial G^k(x, y)}{\partial v(x)} (E_\alpha(y) - E_0(y)) \, ds(y) = \sum_{j=1}^m \left(1 - \frac{\mu_0}{\mu_j}\right) \int_{z_j+\alpha B_j} \text{grad} E_\alpha(y) \cdot \text{grad}_y \frac{\partial G^k(x, y)}{\partial v(x)} \, dy + k^2 \sum_{j=1}^m \left(\frac{\varepsilon_j}{\varepsilon_0} - 1\right) \int_{z_j+\alpha B_j} E_\alpha(y) \frac{\partial G^k(x, y)}{\partial v(x)} \, dy, \tag{10}$$

for x on $\partial\Omega$. The expansion in the argument α of the right-hand side integral in (7) or (8) or (10) was carried out in [3]. A formal derivation followed by a rigorous proof for the derivation of the first two terms can be found in that reference. They are of orders 2 and 3 in α , respectively. This is an improvement over [13] where only the first term of the expansion was found; see also the prior works of Cedio-Fengya et al. [6] for the conductivity problem and Friedman and Vogelius [7] for the case of perfectly conducting or insulating inhomogeneities. These asymptotic expansions are designed for developing efficient algorithms to identify dielectric inhomogeneities of small diameter with important applications in medical imaging, detection of breast cancer, tumors, and land mines, see [1,2]. The higher-order terms in these formulas are essential particularly when the leading order term in the asymptotic expansion of the electromagnetic fields vanishes. The asymptotic formula derived in [3] involves polarization tensors associated with the electromagnetic inhomogeneities that seem to be natural generalizations of the tensors that have been introduced by Schiffer and Szegő [12] and thoroughly studied by many other authors, see [6–8,10,11].

In this paper we propose to present numerical results illustrating the use of the first two terms and we show that the term of order 3 vanishes for some geometries of inhomogeneities including ellipses. We also show that usual numerical schemes for finding the term of order 3 preserve the fact that these are zero for symmetric inhomogeneities, in other words if the related computer codes are run for symmetric inhomogeneities, those terms of order 3 are roughly as close to 0 as machine precision permits even for small values of grid points on the boundary of the inhomogeneities. For the sake of simplicity all numerics for this paper were performed when only one inhomogeneity is present. We are confident that the same results hold when several inhomogeneities are present since all the mathematical arguments hold in that case too. Actually in the case where all inhomogeneities are ellipses numerical experiments related to the terms of order 2 were

already successfully conducted. They were of course missing the existence of those terms of order 3. They indicated that the remainder term in the expansion is of order 4, see [6,14].

We can safely drop the index j in B_j and z_j all through the rest of this paper so that the only inhomogeneity will be B , centered at z and denote the electric permittivity and magnetic permeability inside $z + \alpha B$ by ϵ_* and μ_* , respectively.

Although this paper focuses on TE fields, all the results are easily carried over to the case of TM fields. Neumann problems are solved in place of Dirichlet problems and the roles of ϵ and μ are exchanged.

Remark. An analog expansion within the framework of the three dimensional theory has been developed for the Helmholtz equation as well as for the full Maxwell’s equations, see [1,4]. However in that case the numerical schemes for solving for the electric field throughout a body and for finding numerical values for the correction terms in the expansion, are much more involved and numerically costly. It is noteworthy that highly accurate and fast schemes for solving integral equations on the boundary of three dimensional bodies have just recently been the subject of a state of the art study, see [5].

2. The corrections of orders 2 and 3

In this section we give explicit formulas for the terms in α^2 and α^3 appearing when expanding the right-hand side of (7) and of (10).

Let $\hat{v}_{1i}^\mu(\xi_1, \xi_2)$, $i = 1, 2$, be the unique solution to

$$\begin{cases} \operatorname{div} \frac{1}{\mu} \operatorname{grad} \hat{v}_{1i}^\mu = 0, & \text{in } \mathbb{R}^2, \\ \lim_{|\xi| \rightarrow +\infty} (\hat{v}_{1i}^\mu - \xi_i) = 0. \end{cases} \tag{11}$$

Eq. (11) contains implicitly the conditions on \hat{v}_{1i}^μ across the boundary ∂B . \hat{v}_{1i}^μ is required to be continuous across ∂B and its normal derivative has to satisfy the following jump condition:

$$\frac{1}{\mu_0} \frac{\hat{v}_{1i}^\mu}{v} \Big|_+ - \frac{1}{\mu_*} \frac{\hat{v}_{1i}^\mu}{v} \Big|_- = 0.$$

Existence and uniqueness for \hat{v}_{1i}^μ will be outlined in a subsequent section. The expression of the correction term of order 2 corresponding to the expansion of the right-hand side in (7) is, as rigorously derived in [3,13],

$$E_{1,D}(x) = \left(1 - \frac{\mu_0}{\mu_*}\right) \partial_i E_0(z) M_{ij}^{1,1} \cdot \partial_{z_j} G_D(x, z) + k^2 \left(1 - \frac{\epsilon_*}{\epsilon_0}\right) |B| E_0(z) G_D(x, z), \tag{12}$$

where the generalized polarization tensor $M^{1,1}$ of order (1, 1) is defined by

$$M_{ij}^{1,1} = \int_{\partial B} \frac{\partial \hat{v}_{1i}^\mu}{\partial v} \Big|_- (\xi) \xi_j \, ds(\xi). \tag{13}$$

Analog expressions are obtained for the expansions of the right-hand sides of (8) and (10). In expanding the integrals in (10) the term of order 2 is

$$\frac{\partial E_{1,k}(x)}{\partial v(x)} = \left(1 - \frac{\mu_0}{\mu_*}\right) \partial_i E_0(z) M_{ij}^{1,1} \cdot \partial_j \frac{\partial G^k(x, z)}{\partial v(x)} + k^2 \left(1 - \frac{\epsilon_*}{\epsilon_0}\right) |B| E_0(z) \frac{\partial G^k(x, z)}{\partial v(x)}. \tag{14}$$

We obtain from [3] the two formulas

$$E_\alpha(x) - E_0(x) - \alpha^2 E_{1,D}(x) = O(\alpha^3), \tag{15}$$

for x in Ω bounded away from B ,

$$\frac{1}{2} \left(\frac{\partial E_\alpha(x)}{\partial v(x)} - \frac{\partial E_0(x)}{\partial v(x)} \right) - \int_{\partial\Omega} \frac{\partial G^k(x,y)}{\partial v(x)} (E_\alpha(y) - E_0(y)) \, ds(y) - \alpha^2 \frac{\partial E_{1,k}(x)}{\partial v(x)} = O(\alpha^3), \tag{16}$$

for x on $\partial\Omega$. We need to introduce four other functions analog to \hat{v}_{1i}^μ in order to find the next term in the expansions of (7) and (10). Let $\hat{v}_{2ij}^{\mu,\varepsilon}(\zeta_1, \zeta_2)$, $i = 1, 2, j = 1, 2$, be the unique solution to the following problem:

$$\begin{cases} \operatorname{div} \frac{1}{\mu} \operatorname{grad} \hat{v}_{2ij}^{\mu,\varepsilon} = \frac{1}{\mu_0} \frac{\varepsilon(\zeta)}{\varepsilon_0} \delta_{ij} \text{ in } \mathbb{R}^2, \\ \lim_{|\zeta| \rightarrow +\infty} \left(\hat{v}_{2ij}^{\mu,\varepsilon} - \frac{1}{2} \zeta_i \zeta_j - \delta_{ij} \frac{1}{2\pi} |B| \left(1 - \frac{\varepsilon_*}{\varepsilon_0} \right) \log |\zeta| \right) = 0. \end{cases} \tag{17}$$

Eq. (17) contains implicitly the conditions on $\hat{v}_{2ij}^{\mu,\varepsilon}$ across the boundary ∂B . $\hat{v}_{2ij}^{\mu,\varepsilon}$ is required to be continuous across ∂B and its normal derivative has to satisfy the following jump condition:

$$\frac{1}{\mu_0} \frac{\hat{v}_{2ij}^{\mu,\varepsilon}}{v} \Big|_+ - \frac{1}{\mu_*} \frac{\hat{v}_{2ij}^{\mu,\varepsilon}}{v} \Big|_- = 0.$$

Existence and uniqueness for $\hat{v}_{2ij}^{\mu,\varepsilon}$ will be outlined in a following section. In order to be consistent with the notations introduced in [3], we define the generalized polarization tensors of order (1, 2) and (2, 1) by

$$M_{ijk}^{1,2} = \frac{1}{2} \int_{\partial B} \frac{\partial \hat{v}_{1i}^\mu}{\partial v} \Big|_- (\zeta) \zeta_j \zeta_k \, ds(\zeta), \tag{18}$$

and

$$M_{ijk}^{2,1} = \int_{\partial B} \frac{\partial \hat{v}_{2ij}^{\mu,\varepsilon}}{\partial v} \Big|_- (\zeta) \zeta_k \, ds(\zeta) - \delta_{ij} \frac{\varepsilon_* \mu_*}{\varepsilon_0 \mu_0} \int_B \zeta_k \, d\zeta. \tag{19}$$

The expression of the correction term of order 3 corresponding to the expansion of the right-hand side of (7) is

$$\begin{aligned} E_{2,D}(x) = & \left(1 - \frac{\mu_0}{\mu_*} \right) \left[\partial_{ij}^2 E_0(z) M_{ijk}^{2,1} \partial_{k,z} G_D(x, z) + \partial_i E_0(z) M_{ijk}^{1,2} \partial_{jk,z}^2 G_D(x, z) \right] \\ & + k^2 \left(\frac{\varepsilon_*}{\varepsilon_0} - 1 \right) \left[\partial_i E_0(z) \left(\int_B \hat{v}_{1i}^\mu(\zeta) \, d\zeta \right) G_D(x, z) + E_0(z) \left(\int_B \zeta_j \, d\zeta \right) \partial_{j,z} G_D(x, z) \right]. \end{aligned} \tag{20}$$

Analog expressions are obtained for the expansions of the right-hand sides of (8) and (10). In expanding the integrals in (10) the term of order 3 is

$$\begin{aligned} \frac{\partial E_{2,k}(x)}{\partial v(x)} = & \left(1 - \frac{\mu_0}{\mu_*} \right) \left[\partial_{ij}^2 E_0(z) M_{ijk}^{2,1} \partial_{k,z} \frac{\partial G^k(x, z)}{\partial v(x)} + \partial_i E_0(z) M_{ijk}^{1,2} \partial_{jk,z}^2 \frac{\partial G^k(x, z)}{\partial v(x)} \right] \\ & + k^2 \left(\frac{\varepsilon_*}{\varepsilon_0} - 1 \right) \left[\partial_i E_0(z) \left(\int_B \hat{v}_{1i}^\mu(\zeta) \, d\zeta \right) \frac{\partial G^k(x, z)}{\partial v(x)} + E_0(z) \left(\int_B \zeta_j \, d\zeta \right) \partial_{j,z} \frac{\partial G^k(x, z)}{\partial v(x)} \right]. \end{aligned} \tag{21}$$

We obtain from [3] the two formulas

$$E_x(x) - E_0(x) - \alpha^2 E_{1,D}(x) - \alpha^3 E_{2,D}(x) = O(\alpha^4), \tag{22}$$

for x in Ω bounded away from B , and

$$\frac{1}{2} \left(\frac{\partial E_x(x)}{\partial v(x)} - \frac{\partial E_0(x)}{\partial v(x)} \right) - \int_{\partial\Omega} \frac{\partial G^k(x, y)}{\partial v(x)} (E_x(y) - E_0(y)) ds(y) - \alpha^2 \frac{\partial E_{1,k}(x)}{\partial v(x)} - \alpha^3 \frac{\partial E_{2,k}(x)}{\partial v(x)} = O(\alpha^4), \tag{23}$$

for x on $\partial\Omega$.

3. On the existence and uniqueness of the functions \hat{v}_{1i}^μ and $\hat{v}_{2ij}^{\mu,\varepsilon}$

Using the strong maximum principle for harmonic equations and the fact that the limits at infinity in (11) and in (17) are to be understood in a uniform sense we see that Eqs. (11) and (17) have at most one solution. For the existence of those functions we set

$$\hat{g}_{1i}^\mu = \hat{v}_{1i}^\mu - \zeta_i, \quad \hat{g}_{2ij}^{\mu,\varepsilon} = \begin{cases} \hat{v}_{2ij}^{\mu,\varepsilon} - \frac{1}{2} \zeta_i \zeta_j - \delta_{ij} \frac{1}{2\pi} |B| \left(1 - \frac{\varepsilon}{\varepsilon_0}\right) \log(|\zeta|) & \text{in } \mathbb{R}^2 \setminus \bar{B}, \\ \hat{v}_{2ij}^{\mu,\varepsilon} - \frac{1}{2} \delta_{ij} \frac{\mu_* \varepsilon_*}{\mu_0 \varepsilon_0} \zeta_i \zeta_j & \text{in } B. \end{cases} \tag{24}$$

That way \hat{g}_{1i}^μ and $\hat{g}_{2ij}^{\mu,\varepsilon}$ are harmonic functions in $\mathbb{R}^2 \setminus \bar{B}$ and in B that tend uniformly to 0 at infinity. Then \hat{g}_{1i}^μ can be sought as a single layer potential and $\hat{g}_{2ij}^{\mu,\varepsilon}$ can be sought as the sum of a single and a double layer potential. Integral equations on ∂B have to be solved in the unknown densities, they are of Fredholm type. Well posedness for these equations can be derived from the uniqueness for \hat{g}_{1i}^μ and $\hat{g}_{2ij}^{\mu,\varepsilon}$.

In order to numerically evaluate (13), (18), and (19) it will prove useful to solve for $\hat{v}_{1i}^\mu, \frac{\partial \hat{v}_{1i}^\mu}{\partial v} \Big|_-$ and $\hat{v}_{2ij}^{\mu,\varepsilon}$ restricted to ∂B . They respectively satisfy the following equations which are derived from representing \hat{g}_{1i}^μ and $\hat{g}_{2ij}^{\mu,\varepsilon}$ using integrals on ∂B . In the following equations:

$$G^0(\zeta, \zeta') = \frac{1}{2\pi} \log \left(\frac{1}{|\zeta - \zeta'|} \right)$$

is the free space Green’s function for the laplacian and ζ is any point on ∂B . We have

$$\frac{1}{2} \left(1 + \frac{\mu_0}{\mu_*} \right) \hat{v}_{1i}^\mu(\zeta) + \left(\frac{\mu_0}{\mu_*} - 1 \right) \int_{\partial B} \hat{v}_{1i}^\mu(\zeta') \frac{\partial G^0}{\partial v(\zeta')}(\zeta, \zeta') ds(\zeta') = \zeta_i, \tag{25}$$

$$\frac{1}{2} \left(1 + \frac{\mu_0}{\mu_*} \right) \frac{\partial \hat{v}_{1i}^\mu}{\partial v} \Big|_- (\zeta) + \left(\frac{\mu_0}{\mu_*} - 1 \right) \int_{\partial B} \frac{\partial \hat{v}_{1i}^\mu}{\partial v} \Big|_- (\zeta') \frac{\partial G^0}{\partial v(\zeta)}(\zeta, \zeta') ds(\zeta') = v_i, \tag{26}$$

and

$$\frac{1}{2} \left(1 + \frac{\mu_0}{\mu_*} \right) \hat{v}_{2ij}^{\mu,\varepsilon} \Big|_{\partial B}(\zeta) + \left(\frac{\mu_0}{\mu_*} - 1 \right) \int_{\partial B} \hat{v}_{2ij}^{\mu,\varepsilon} \Big|_{\partial B}(\zeta') \frac{\partial G^0}{\partial v(\zeta')}(\zeta, \zeta') ds(\zeta') = \frac{1}{2} \zeta_i \zeta_j + \delta_{ij} \left(1 - \frac{\varepsilon_*}{\varepsilon_0} \right) \int_B G^0(\zeta, \zeta') d\zeta'. \tag{27}$$

The above equation can be rewritten in terms of integrals on ∂B only. Indeed assume in a first step that ζ is in $\mathbb{R}^2 \setminus \bar{\Omega}$ and integrate by parts

$$\begin{aligned} \int_B G^0(\xi, \xi') d\xi' &= \int_{\partial B} G^0(\xi, \xi') \frac{1}{2} \frac{\partial(\xi_1')^2}{\partial v(\xi')} ds(\xi') - \int_B \nabla_{\xi'} G^0(\xi, \xi') \nabla \left(\frac{1}{2} (\xi_1')^2 \right) d\xi' \\ &= \int_{\partial B} G^0(\xi, \xi') \xi_1' v_1(\xi') ds(\xi') - \int_{\partial B} \frac{1}{2} (\xi_1')^2 \frac{\partial G^0}{\partial v(\xi')}(\xi, \xi') ds(\xi'). \end{aligned} \tag{28}$$

As ξ approaches ∂B we obtain at the limit

$$\int_B G^0(\xi, \xi') d\xi' = \int_{\partial B} G^0(\xi, \xi') \xi_1' v_1(\xi') ds(\xi') ds(\xi') - \int_{\partial B} \frac{1}{2} (\xi_1')^2 \frac{\partial G^0}{\partial v(\xi')}(\xi, \xi') ds(\xi') - \frac{1}{4} (\xi_1')^2.$$

Modifying the right-hand side in (27) accordingly, we obtain

$$\begin{aligned} &\frac{1}{2} \left(1 + \frac{\mu_0}{\mu_*} \right) \hat{v}_{2ij}^{\mu, \epsilon} |_{\partial B}(\xi) + \left(\frac{\mu_0}{\mu_*} - 1 \right) \int_{\partial B} \hat{v}_{2ij}^{\mu, \epsilon} |_{\partial B}(\xi') \frac{\partial G^0}{\partial v(\xi')}(\xi, \xi') ds(\xi') \\ &= \frac{1}{2} \xi_i \xi_j + \delta_{ij} \left(1 - \frac{\epsilon_*}{\epsilon_0} \right) \left(\int_{\partial B} G^0(\xi, \xi') \xi_1' v_1(\xi') d\xi' - \int_{\partial B} \frac{\partial G^0}{\partial v(\xi')}(\xi, \xi') \frac{(\xi_1')^2}{2} d\xi' - \frac{(\xi_1')^2}{4} \right). \end{aligned} \tag{29}$$

4. Numerical results

To illustrate the above expansions, we solved numerical equations for three different geometries for the inhomogeneity B . We used Nystrom’s method for integral equations with $2n$ grid points on $\partial\Omega$ and $2n$ grid points on ∂B with $n = 90$ although in many cases quite accurate results can be obtained with smaller values of n . The numerical method we developed is, in its outline, inspired by chapter 12 from [9]. However the method described in that reference corresponds to the trivial case where no inhomogeneity is present. We had to extend that method to domains with inclusions and therefore we had to solve systems of integral equations. A detailed account of the set up of the integral equations for a domain with inclusions and the discretization of those systems of equations can be found in [14]. In this paper we discuss the details of the numerical schemes designed to obtain the corrections of order 2 and 3. This will be done in the following section. For each of the three different geometries we solved for 11 values of α taking $z + \alpha B$ to be the only inhomogeneity in Ω . In all cases Ω is the unit circle of \mathbb{R}^2 , $z = (z_1, z_2)$ is the point $(0.2, 0.3)$ and the values for α are $10^{-(j-1)/10}$, $1 \leq j \leq 11$. The values for the electromagnetic coefficients were chosen to be $k = 2$, $k_* = 1.55 + 2i$, $\mu = .5$, $\mu_* = .725$. Note that although ϵ and ϵ_* are not specified this data provides a unique value for the ratio ϵ_*/ϵ via the formula

$$\frac{\epsilon_*}{\epsilon} = \frac{k_*^2 \mu}{k^2 \mu_*}. \tag{30}$$

We choose to impose the following Dirichlet boundary condition on $\partial\Omega$

$$E_x(x_1, x_2) = H_0^1(k|(x_1, x_2) - (q, 0)|),$$

where $q = 4$ and H_0^1 is the usual Hankel function of the first kind of order 0. In the first case ∂B is the ellipse of equation

$$\begin{cases} x(t) = a \cos(\theta) \cos(t) - b \sin(\theta) \sin(t) + z_1, \\ y(t) = a \sin(\theta) \cos(t) + b \cos(\theta) \sin(t) + z_2, \end{cases} \tag{31}$$

for $0 \leq t \leq 2\pi$ where $\theta = 1$, $a = 0.08$, $b = 0.06$. Ω and B are sketched in Fig. 1.

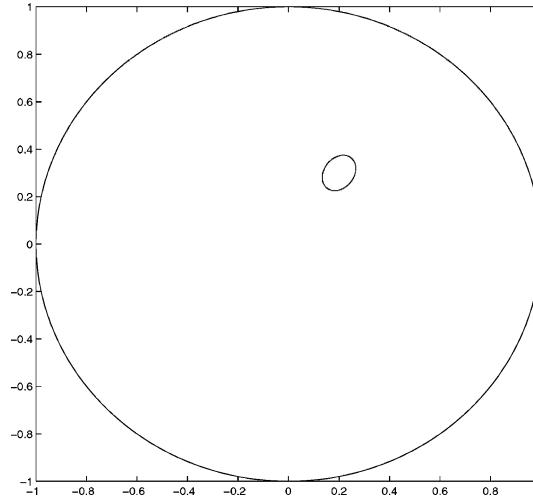


Fig. 1. The domain Ω and the inhomogeneity shaped as an ellipse, $\alpha = 1$.

In Fig. 2, we plotted using stars the log of the discrete maximum of the magnitude of the left-hand side of (16) against the log of α . The dashed line is just a line of slope 4. We plotted it to give a sense of scale.

This graphic demonstrates that the quantity in the left-hand side of (16) is of order α^4 although it was expected to be of order α^3 . In the following table we wrote the maximum of the computed value of the magnitude of $M_{ijk}^{2,1}$ as i, j, k run over $\{1, 2\}$ for different values of n where $2n$ is the number of grid points on ∂B used for calculating $M_{ijk}^{2,1}$. In the last column we indicate the discrete maximum of the magnitude of the left-hand side of (16) in the case where $\alpha = 1$.

Similar orders of magnitude are observed for

$$M_{ijk}^{1,2}, \quad \int_B \hat{v}_{1i}^\mu(\xi) d\xi \quad \text{and} \quad \int_B \xi_j d\xi.$$

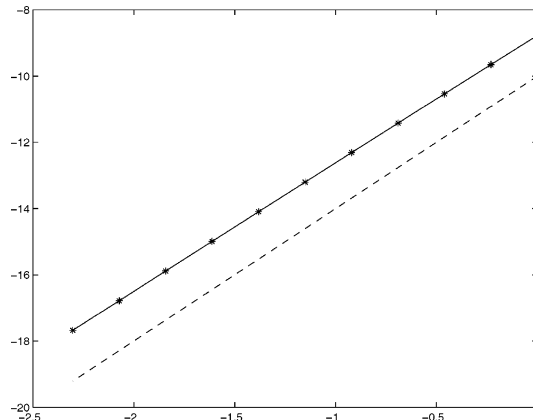


Fig. 2. Numerical results for the geometry sketched in Fig. 1. The log of α is on the horizontal axis, the dashed is a line of slope 4. The 11 crosses represent the log of the norm of (16) graphed against the log of α .

It is apparent that our code preserves the fact that $\partial E_{2,k}/\partial v$ is zero if B is an ellipse. This will be rigorously proved in the next section.

In the second case B is the object shaped as a kidney bean whose contour is given by the equations

$$\begin{cases} x(t) = a \cos(\theta) \sqrt{1.1 + \cos(t)} \cos(t) - b \sin(\theta) \sqrt{1.1 + \cos(t)} \sin(t) + z_1, \\ y(t) = a \sin(\theta) \sqrt{1.1 + \cos(t)} \cos(t) + b \cos(\theta) \sqrt{1.1 + \cos(t)} \sin(t) + z_2, \end{cases} \quad (32)$$

for $0 \leq t \leq 2\pi$ and where $\theta = 1.4$, $a = 0.08$, $b = 0.06$. Ω and B are sketched in Fig. 3.

A magnified picture of the object B is represented in Fig. 4.

In Fig. 5, we plotted using stars the log of the discrete maximum of the magnitude of the quantity (16) against the log of α . The dotted line is just a line of slope 3. We also plotted using squares the log of the discrete maximum of the magnitude of the quantity in (23) against the log of α . The dashed line is just a line of slope 4.

This graphic demonstrates that the expression in (16) is of order α^3 whereas the expression in (23) is of order α^4 .

In this last case B is the object whose shape is sketched in Fig. 6 and whose contour is given by the equations

$$\begin{cases} x(t) = a \cos(\theta) \sin(\cos(t)) - b \sin(\theta) \sin(\sin(t)) + z_1, \\ y(t) = a \sin(\theta) \sin(\cos(t)) + b \cos(\theta) \sin(\sin(t)) + z_2, \end{cases} \quad (33)$$

for $0 \leq t \leq 2\pi$ and where $\theta = -2$, $a = 0.08$, $b = 0.06$. Ω and B are sketched in Fig. 7.

In Fig. 8, we plotted using stars the log of the discrete maximum of the magnitude of the expression (16) against the log of α . The dashed line is a line of slope 4.

This graphic demonstrates that the expression (16) is of order α^4 . The same remark holds as in the first case the quantities

$$M_{ijk}^{1,2}, \quad M_{ijk}^{2,1}, \quad \int_B \hat{v}_{1i}^\mu(\xi) d\xi \quad \text{and} \quad \int_B \xi_j d\xi$$

are of negligible magnitude even for small values of n .

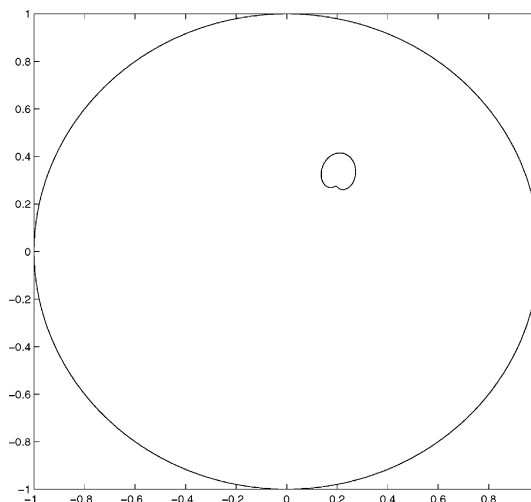


Fig. 3. The inhomogeneity defined by (32) in Ω , $\alpha = 1$.

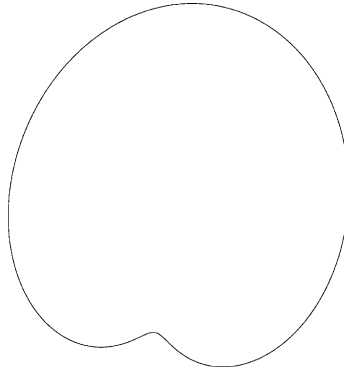


Fig. 4. The inhomogeneity defined by (32) has no center of symmetry.

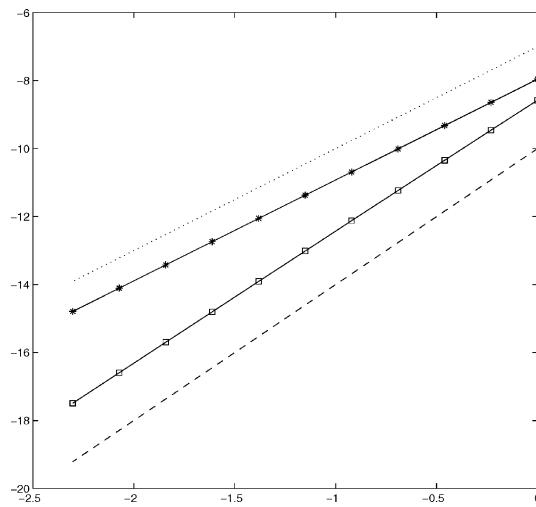


Fig. 5. Numerical results for the geometry sketched in Fig. 3. The log of α is on the horizontal axis. The dashed line is of slope 4, the dotted line is of slope 3. The 11 squares represent the log of the norm of (16) graphed against the log of α . The 11 crosses represent the log of the norm of (23) graphed against the log of α .

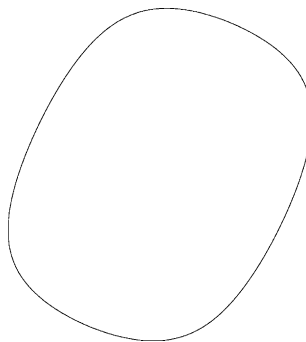


Fig. 6. The inhomogeneity defined by (33) is symmetric about its center.

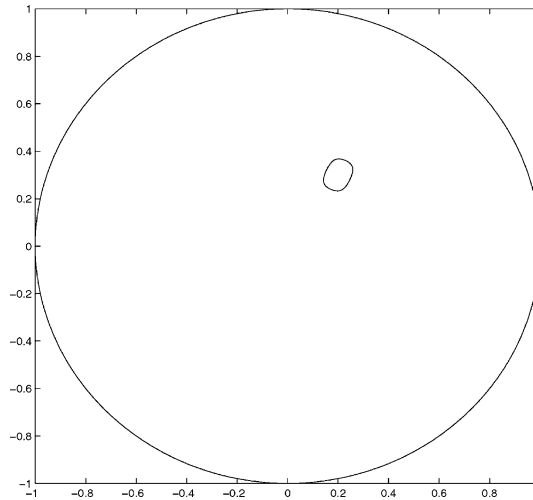


Fig. 7. The inhomogeneity defined by (33) in Ω , $\alpha = 1$.

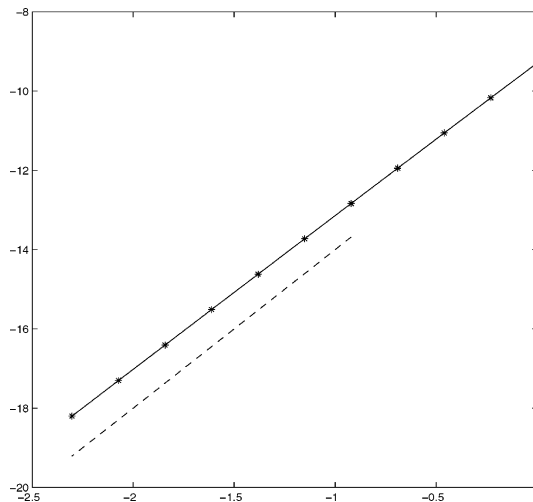


Fig. 8. Numerical results for the geometry sketched in Fig. 7. The log of α is on the horizontal axis. The dashed line is of slope 4. The 11 crosses represent the log of the norm of (16) graphed against the log of α .

In all cases we also evaluated the normal derivative on $\partial\Omega$ of the expressions in (15) and (22). All the numerical output was of the same order of magnitude as in (16) and (23), as expected.

5. Study of the terms of order 3 for inhomogeneities that are symmetric about their center

In this section we show that the terms of order 3 vanish for inhomogeneities that are symmetric about a point. For convenience we denote B the fixed inhomogeneity and we suppose that $B = -B$. We first derive

Table 1

n	$\max M_{ijk}^{2,1} $	max norm of (16)
1	2.206191581974134E – 019	1.49262741086771
2	3.255428736223880E – 019	0.537008508295939
4	1.638959506095633E – 019	7.218122917393703E – 002
8	1.521644651974750E – 019	1.024928358871951E – 003
16	3.460914846154340E – 019	1.536839894508883E – 004
32	2.270526284636852E – 019	1.548371599404250E – 004
64	5.086559795829020E – 019	1.548371599403280E – 004

from the definition of \hat{v}_{1i}^{μ} and $\hat{v}_{2ij}^{\mu,\varepsilon}$ that a central symmetry for B implies that those terms of order 3 are zero. Then, in order to explain the magnitude ratio pattern in Table 1, we analyze our computational scheme and prove that it preserves at the discrete level the fact that a central symmetry in B yields zero for the terms of order 3.

5.1. *The continuous case*

Proposition 5.1. *The corrections of order 3 (20) and (21) are zero if the domain B is symmetric about its center. More precisely, each of the tensor terms $M_{ijk}^{1,2}$ and $M_{ijk}^{2,1}$ and each of the integrals*

$$\int_B \hat{v}_{1i}^{\mu}(\xi) d\xi \quad \text{and} \quad \int_B \xi_j d\xi, \quad 1 \leq i, j, k \leq 2,$$

are zero if the domain B is symmetric about its center.

Proof. Without loss of generality we can assume that B is centered at the origin. We first notice that the symmetry assumption for B implies that the functions μ and ε are even. Set $\hat{w}_{1i}^{\mu}(\xi) = \hat{v}_{1i}^{\mu}(-\xi)$. Then the function $\hat{w}_{1i}^{\mu}(\xi)$ satisfies

$$\begin{cases} \operatorname{div} \frac{1}{\mu} \operatorname{grad} \hat{w}_{1i}^{\mu} = 0 \text{ in } \mathbb{R}^2, \\ \lim_{|\xi| \rightarrow +\infty} (\hat{w}_{1i}^{\mu} + \xi_i) = 0. \end{cases} \tag{34}$$

It follows that $\hat{w}_{1i}^{\mu}(\xi) = -\hat{v}_{1i}^{\mu}(\xi)$, that is, \hat{v}_{1i}^{μ} is odd. Set $\hat{w}_{2ij}^{\mu,\varepsilon}(\xi) = \hat{v}_{2ij}^{\mu,\varepsilon}(-\xi)$. $\hat{w}_{2ij}^{\mu,\varepsilon}$ satisfies

$$\begin{cases} \operatorname{div} \frac{1}{\mu} \operatorname{grad} \hat{w}_{2ij}^{\mu,\varepsilon} = \frac{1}{\mu_0} \frac{\varepsilon(\xi)}{\varepsilon_0} \delta_{ij} \text{ in } \mathbb{R}^2, \\ \lim_{|\xi| \rightarrow +\infty} \left(\hat{w}_{2ij}^{\mu,\varepsilon} - \frac{1}{2} \xi_i \xi_j - \delta_{ij} \frac{1}{2\pi} |B| \left(1 - \frac{\varepsilon_0}{\varepsilon} \right) \log |\xi| \right) = 0. \end{cases} \tag{35}$$

It follows that $\hat{v}_{2ij}^{\mu,\varepsilon}$ is even. Next for ξ on ∂B we infer that

$$\frac{\partial \hat{v}_{1i}^{\mu}}{\partial \mathbf{v}} \Big|_{-}(\xi) = -\frac{\partial \hat{v}_{1i}^{\mu}}{\partial \mathbf{v}} \Big|_{-}(-\xi) \quad \text{and} \quad \frac{\partial \hat{v}_{2ij}^{\mu,\varepsilon}}{\partial \mathbf{v}} \Big|_{-}(\xi) = \frac{\partial \hat{v}_{2ij}^{\mu,\varepsilon}}{\partial \mathbf{v}} \Big|_{-}(-\xi). \tag{36}$$

Thus the quantities

$$\frac{\partial \hat{v}_{1i}^{\mu}}{\partial \mathbf{v}} \Big|_{-}(\xi) \xi_j \xi_k \quad \text{and} \quad \frac{\partial \hat{v}_{2ij}^{\mu,\varepsilon}}{\partial \mathbf{v}} \Big|_{-}(\xi) \xi_k$$

are odd for ξ on ∂B . Consequently the change of variables $\xi \rightarrow -\xi$ will prove that $M_{ijk}^{1,2}$ and $M_{ijk}^{2,1}$ as defined in (18) and (19) are zero since ∂B is symmetric about the origin. By the same token we notice that

$$\int_B \hat{v}_{1i}^\mu(\xi) d\xi \quad \text{and} \quad \int_B \xi_j d\xi$$

are zero. We have thus proved that the four terms involved in the definition of the terms order 3 are equal to zero. \square

5.2. Discretizing the equations for the functions \hat{v}_{1i}^μ , $\frac{\partial \hat{v}_{1i}^\mu}{\partial v} \Big|_-$ and $\hat{v}_{2ij}^{\mu,\varepsilon}$ restricted to ∂B

In this section we assume that

$$\text{the contour } \partial B \text{ is given by } (x(t), y(t)), \quad 0 \leq t < 2\pi, \tag{37}$$

where x and y are two real analytic 2π periodic functions. We are first going to describe how to derive a natural discrete scheme for the Eqs. (25), (26) and (29). The derivation of such schemes is a classical exercise and has been performed by many authors. Our reason for giving a detailed formulation of these schemes is merely for the sake of introducing notations that will be indispensable for stating and proving our main result. It is also worth mentioning that these schemes provide an exponentially fast convergence to the solutions of the associated continuous solutions, see [9], in particular chapters 10 and 12, and chapter 14 for a study of numerical stability. If B is symmetric about its center we require the following on x and y :

$$x(t + \pi) = -x(t), \quad y(t + \pi) = -y(t). \tag{38}$$

We start with computing \hat{v}_{1i}^μ in Eq. (25). We denote $K_1(t, s)$ the value of

$$\frac{\partial G^0}{\partial v(\xi')}(\xi, \xi') \text{ at } \xi = (x(t), y(t)), \quad \xi' = (x(s), y(s)).$$

We notice that for a fixed t in $[0, 2\pi]$, the function $s \rightarrow K_1(t, s)$ is real analytic. A calculation yields

$$K_1(t, s) = \begin{cases} -\frac{1}{2\pi} \frac{x'(s)(y(s)-y(t))-y'(s)(x(s)-x(t))}{((x(t)-x(s))^2+(y(t)-y(s))^2)(x'(s)^2+y'(s)^2)^{1/2}} & \text{if } s \neq t, \\ -\frac{1}{2\pi} \frac{1}{2} \frac{x''(t)y'(t)-x'(t)y''(t)}{(x'(t)^2+y'(t)^2)^{3/2}} & \text{if } s = t. \end{cases} \tag{39}$$

The second line in (39) is the limit of the first line as s approaches a fixed t . The length element relative to the chosen parametrization of ∂B is $\sigma(s) = (x'(s)^2 + y'(s)^2)^{1/2}$. Setting $\hat{v}_{1i}^\mu(x(t), y(t)) = v_{1i}(t)$, $i = 1, 2$, $\xi_1(t) = x(t)$ and $\xi_2(t) = y(t)$ we need to solve the integral equation

$$\frac{1}{2} \left(1 + \frac{\mu_0}{\mu_*} \right) v_{1i}(t) + \left(\frac{\mu_0}{\mu_*} - 1 \right) \int_0^{2\pi} K_1(t, s) v_{1i}(s) \sigma(s) ds = \xi_i(t), \quad 0 \leq t \leq 2\pi. \tag{40}$$

We set a uniform grid on $[0, 2\pi]$:

$$t_m = s_m = \frac{m\pi}{n}, \quad m = 0, \dots, 2n - 1. \tag{41}$$

The above equation is simply discretized as

$$\frac{1}{2} \left(1 + \frac{\mu_0}{\mu_*} \right) v_{1i}(t_m) + \left(\frac{\mu_0}{\mu_*} - 1 \right) \frac{\pi}{n} \sum_{l=0}^{2n-1} K_1(t_m, s_l) v_{1i}(s_l) \sigma(s_l) = \xi_i(t_m), \quad m = 0, \dots, 2n - 1. \tag{42}$$

We then focus on computing $\left. \frac{\partial \hat{v}_{1i}^\mu}{\partial v} \right|_-$, that is, we want to find a numerical solution to (26). We denote $K_2(t, s)$ the value of

$$\frac{\partial G^0}{\partial v(\xi)}(\xi, \xi') \text{ at } \xi = (x(t), y(t)), \quad \xi' = (x(s), y(s)).$$

A calculation yields

$$K_2(t, s) = \begin{cases} -\frac{1}{2\pi} \frac{x'(t)(y(t)-y(s))-y'(t)(x(t)-x(s))}{((x(t)-x(s))^2+(y(t)-y(s))^2)(x'(s)^2+y'(s)^2)^{\frac{3}{2}}} & \text{if } s \neq t, \\ -\frac{1}{2\pi} \frac{1}{2} \frac{x''(t)y'(t)-x'(t)y''(t)}{(x'(t)^2+y'(t)^2)^{\frac{3}{2}}} & \text{if } s = t. \end{cases} \tag{43}$$

This makes $s \rightarrow K_2(t, s)$ a smooth function when t is fixed. We set

$$\left. \frac{\partial \hat{v}_{1i}^\mu}{\partial v} \right|_-(x(t), y(t)) = \mathcal{W}_{1i}(t), \quad i = 1, 2,$$

and write the right-hand side of (26) in a more explicit fashion

$$\frac{\partial \xi_i}{\partial v}(x(t), y(t)) = \begin{cases} v_1(t) = -y'(t)/(x'(t)^2 + y'(t)^2)^{1/2} & \text{if } i = 1, \\ v_2(t) = x'(t)/(x'(t)^2 + y'(t)^2)^{1/2} & \text{if } i = 2, \end{cases}$$

where (v_1, v_2) are the coordinates of the exterior normal vector v . We need to solve the integral equation

$$\frac{1}{2} \left(1 + \frac{\mu_0}{\mu_*} \right) \mathcal{W}_{1i}(t) + \left(\frac{\mu_0}{\mu_*} - 1 \right) \int_0^{2\pi} K_2(t, s) \mathcal{W}_{1i}(s) \sigma(s) ds = v_i(t), \quad 0 \leq t \leq 2\pi. \tag{44}$$

The above equation is simply discretized as

$$\frac{1}{2} \left(1 + \frac{\mu_0}{\mu_*} \right) \mathcal{W}_{1i}(t_m) + \left(\frac{\mu_0}{\mu_*} - 1 \right) \frac{\pi}{n} \sum_{l=0}^{2n-1} K_2(t_m, s_l) \mathcal{W}_{1i}(s_l) \sigma(s_l) = v_i(t_m), \quad m = 0, \dots, 2n - 1. \tag{45}$$

We now want to solve for $\hat{v}_{2ij}^{\mu, \epsilon}$. Discretizing Eq. (29) involves more work since the right hand side of (29) contains an integral with a logarithmic integration kernel. However the logarithmic singularity and the integrand are 2π periodic and the integrand is smooth. It proves efficient to follow Nystrom’s integration method as demonstrated in [9]. We set at $\xi = (x(t), y(t)), \xi' = (x(s), y(s)), t \neq s$

$$G^0(\xi, \xi') = -\frac{1}{2\pi} \left(\ln \left(4 \sin^2 \left(\frac{t-s}{2} \right) \right) + K_3(t, s) \right), \tag{46}$$

where K_3 is the smooth function in s when t is fixed

$$K_3(t, s) = \begin{cases} \frac{1}{2} \log \left(\frac{(x(t)-x(s))^2+(y(t)-y(s))^2}{4 \sin^2 \left(\frac{t-s}{2} \right)} \right) & \text{if } s \neq t, \\ \frac{1}{2} \log \left(x'(t)^2 + y'(t)^2 \right) & \text{if } s = t. \end{cases} \tag{47}$$

Setting $\hat{v}_{2ij}^{\mu, \epsilon}(x(t), y(t)) = v_{2ij}(t), i, j = 1, 2$, we need to solve the integral equation

$$\frac{1}{2} \left(1 + \frac{\mu_0}{\mu_*} \right) v_{2ij}(t) + \left(\frac{\mu_0}{\mu_*} - 1 \right) \int_0^{2\pi} K_1(t, s) v_{2ij}(s) \sigma(s) ds = b_{ij}(t), \quad 0 \leq t \leq 2\pi, \tag{48}$$

where the right-hand side b_{ij} is defined as

$$\begin{aligned}
 b_{ij}(t) = & \frac{1}{2} \zeta_i(t) \zeta_j(t) + \delta_{ij} \left(1 - \frac{\varepsilon_*}{\varepsilon_0} \right) \left(\int_0^{2\pi} -\frac{1}{2\pi} \ln \left(4 \sin^2 \left(\frac{t-s}{2} \right) \right) \zeta_1(s) v_1(s) \sigma(s) \, ds \right. \\
 & \left. + \int_0^{2\pi} -\frac{1}{2\pi} K_3(t, s) \zeta_1(s) v_1(s) \sigma(s) \, ds - \int_0^{2\pi} K_1(t, s) \frac{(\zeta_1(s))^2}{2} \sigma(s) \, ds - \frac{(\zeta_1(t))^2}{4} \right). \tag{49}
 \end{aligned}$$

Eq. (48) is discretized as follows

$$\frac{1}{2} \left(1 + \frac{\mu_0}{\mu_*} \right) v_{2ij}(t_m) + \left(\frac{\mu_0}{\mu_*} - 1 \right) \frac{\pi}{n} \sum_{l=0}^{2n-1} K_1(t_m, s_l) v_{2ij}(s_l) \sigma(s_l) = b_{ij}(t_m), \quad m = 0, \dots, 2n - 1. \tag{50}$$

To find $b_{ij}(t_m)$ we apply the following method, see [9]:

$$\begin{aligned}
 b_{ij}(t_m) = & \frac{1}{2} \zeta_i(t_m) \zeta_j(t_m) - \delta_{ij} \left(1 - \frac{\varepsilon_*}{\varepsilon_0} \right) \left(\sum_{l=0}^{2n-1} Q(m, l) \zeta_1(s_l) v_1(s_l) \sigma(s_l) + \frac{1}{2n} \sum_{l=0}^{2n-1} K_3(t_m, s_l) \zeta_1(s_l) v_1(s_l) \sigma(s_l) \right. \\
 & \left. + \frac{\pi}{n} \sum_{l=0}^{2n-1} K_1(t_m, s_l) \frac{(\zeta_1(s_l))^2}{2} \sigma(s_l) + \frac{(\zeta_1(t_m))^2}{4} \right), \tag{51}
 \end{aligned}$$

where

$$Q(m, l) = -\frac{1}{n} \left[\sum_{r=1}^{n-1} \frac{1}{r} \cos r(t_m - t_l) + \frac{1}{2n} \cos n(t_m - t_l) \right]. \tag{52}$$

5.3. Main result

The theory guarantees that the linear systems (42), (45), (50)/(51) respectively in the unknowns $v_{1i}(t_m)$, $\mathcal{W}_{1i}(t_m)$, $v_{2ij}(t_m)$ for $m = 0, \dots, 2n - 1$ are well posed for n large enough. They converge exponentially to the solution of the corresponding continuous problems provided that $x(t)$ and $y(t)$ are real analytic functions of t . In addition we are going to prove that for objects B symmetric about their center the following result holds.

Proposition 5.2. *Suppose that x and y are smooth, 2π periodic and satisfy (38). Suppose that the points t_m are given by (41) and that the linear systems (42), (45), (50)/(51) in their respective unknowns $v_{1i}(t_m)$, $\mathcal{W}_{1i}(t_m)$, $v_{2ij}(t_m)$ for $m = 0, \dots, 2n - 1$ are regular. Then for $m = 0, \dots, n - 1$, we have*

$$\begin{aligned}
 v_{1i}(t_{m+n}) &= -v_{1i}(t_m), \\
 \mathcal{W}_{1i}(t_{m+n}) &= -\mathcal{W}_{1i}(t_m), \\
 v_{2ij}(t_{m+n}) &= v_{2ij}(t_m).
 \end{aligned} \tag{53}$$

In other words the numerical schemes that we used preserve the fact that \hat{v}_{1i}^μ and $\left. \frac{\partial \hat{v}_{1i}^\mu}{\partial v} \right|_-$ are odd and that $\hat{v}_{2ij}^{\mu, \varepsilon}$ is even. The following lemma will be the main ingredient in proving Proposition 5.2.

Lemma 5.1. *Let X and C be two $2n \times 1$ vectors and A be a regular $2n \times 2n$ matrix such that $AX = C$. Denote J the $2n \times 2n$ matrix*

$$J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \tag{54}$$

where I_n is the $n \times n$ identity matrix. Assume that J and A commute, that is $JA = AJ$. If $JC + C = 0$, that is C is in the nullspace of $J + I_{2n}$ (respectively $JC - C = 0$, C is in the nullspace of $J - I_{2n}$) then $JX + X = 0$, that is X is also in the nullspace of $J + I_{2n}$ (respectively $JX - X = 0$, X is also in the nullspace of $J - I_{2n}$).

Proof. We notice that $J^2 = I_{2n}$ thus $J^{-1} = J$. It follows that $JA^{-1} = A^{-1}J$. If $JC + C = 0$ then

$$JX + X = JA^{-1}C + A^{-1}C = A^{-1}(JC + C) = 0.$$

The proof for the case $JC - C = 0$ is similar. \square

Proof of Proposition 5.2. We first notice that if we write a $2n \times 2n$ matrix A in blocks

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where each of the blocks is of size $n \times n$, then A and J commute if and only if $A_1 = A_4$ and $A_2 = A_3$. We also notice that the $2n \times 1$ vector $(C_m)_{0 \leq m \leq 2n-1}$ is in the nullspace of $J + I_{2n}$ if and only if $C_{m+n} = -C_m$ for $m = 0, \dots, n-1$ and $(C_m)_{0 \leq m \leq 2n-1}$ is in the nullspace of $J - I_{2n}$ if and only if $C_{m+n} = C_m$ for $m = 0, \dots, n-1$.

Using the assumption (38) on x and y and the definitions (39), (43), (47) for K_1, K_2, K_3 we derive $K_r(t + \pi, s + \pi) = K_r(t, s)$ and $K_r(t + \pi, s) = K_r(t, s + \pi)$ for r equal to 1, 2 or 3. For any l and m in $\{0, \dots, n-1\}$, according to (41), we can write $t_{m+n} = t_m + \pi$ and $s_{l+n} = s_l + \pi$ and thus we have found that

$$K_r(t_{m+n}, s_{l+n}) = K_r(t_m, s_l), \quad K_r(t_{m+n}, s_l) = K_r(t_m, s_{l+n}), \quad \text{where } l, m \in \{0, \dots, n-1\}, \quad r \in \{1, 2, 3\}. \tag{55}$$

In addition we also have for the length element the property $\sigma(s_{l+n}) = \sigma(s_l)$. It follows that each of the matrices derived from the linear systems (42), (45) and (50) commutes with J . Since $\xi_i(t_{m+n}) = -\xi_i(t_m)$ and $v_i(t_{m+n}) = v_i(t_m)$ for $m = 0, \dots, n-1$ each of the right-hand side vector in (42) and (45) is the nullspace of $J + I_{2n}$. We can apply Lemma 5.1 to the linear systems (42) and (45) to obtain the first two identities in (53).

There remains to prove that the right-hand side vector in (50) is in the nullspace of $J - I_{2n}$. Since $t_{m+n} - t_{l+n} = t_m - t_l$, where l and m are in $\{0, \dots, n-1\}$ the definition (52) for Q implies that $Q(m+n, l+n) = Q(m, n)$. In addition Q enjoys the following property for l and m in $\{0, \dots, n-1\}$

$$\begin{aligned} Q(m+n, l) &= -\frac{1}{n} \left[\sum_{r=1}^{n-1} \frac{1}{r} \cos r(t_{m+n} - t_l) + \frac{1}{2n} \cos n(t_{m+n} - t_l) \right] \\ &= \frac{1}{n} \left[\sum_{r=1}^{n-1} \frac{1}{r} \cos(r(m-l)\frac{\pi}{n} + \pi) + \frac{1}{2n} \cos((m+1-l)\pi) \right] \\ &= \frac{1}{n} \left[\sum_{r=1}^{n-1} \frac{1}{r} \cos(r(m-l)\frac{\pi}{n} - \pi) + \frac{1}{2n} \cos((m-(l+1))\pi) \right] \\ &= -\frac{1}{n} \left[\sum_{r=1}^{n-1} \frac{1}{r} \cos r(t_m - t_{l+n}) + \frac{1}{2n} \cos n(t_m - t_{l+n}) \right] = Q(m, l+n). \end{aligned} \tag{56}$$

Thus Q enjoys a property akin to (55), namely,

$$Q(m+n, l+n) = Q(m, n), \quad Q(m+n, l) = Q(m, l+n), \quad l, m \in \{0, \dots, n-1\}. \tag{57}$$

Then using (51), (57) and (55), for m in $\{0, \dots, n - 1\}$

$$\begin{aligned}
 b_{ij}(t_{m+n}) &= \frac{1}{2} \zeta_i(t_{m+n}) \zeta_j(t_{m+n}) - \delta_{ij} \left(1 - \frac{\varepsilon_*}{\varepsilon_0}\right) \left(\sum_{l=0}^{2n-1} Q(m+n, l) \zeta_1(s_l) v_1(s_l) \sigma(s_l) \right. \\
 &\quad + \frac{1}{2n} \sum_{l=0}^{2n-1} K_3(t_{m+n}, s_l) \zeta_1(s_l) v_1(s_l) \sigma(s_l) + \frac{\pi}{n} \sum_{l=0}^{2n-1} K_1(t_{m+n}, s_l) \frac{(\zeta_1(s_l))^2}{2} \sigma(s_l) \\
 &\quad \left. + \frac{(\zeta_1(t_{m+n}))^2}{4} \right) = \frac{1}{2} \zeta_i(t_m) \zeta_j(t_m) - \delta_{ij} \left(1 - \frac{\varepsilon_*}{\varepsilon_0}\right) \left(\sum_{l=0}^{n-1} [Q(m+n, l) \zeta_1(s_l) v_1(s_l) \sigma(s_l) \right. \\
 &\quad + Q(m+n, l+n) \zeta_1(s_{l+n}) v_1(s_{l+n}) \sigma(s_{l+n})] + \frac{1}{2n} \sum_{l=0}^{n-1} [K_3(t_{m+n}, s_l) \zeta_1(s_l) v_1(s_l) \sigma(s_l) \\
 &\quad + K_3(t_{m+n}, s_{l+n}) \zeta_1(s_{l+n}) v_1(s_{l+n}) \sigma(s_{l+n})] + \frac{\pi}{n} \sum_{l=0}^{n-1} [K_1(t_{m+n}, s_l) \frac{(\zeta_1(s_l))^2}{2} \sigma(s_l) \\
 &\quad + K_1(t_{m+n}, s_{l+n}) \frac{(\zeta_1(s_{l+n}))^2}{2} \sigma(s_{l+n})] + \frac{(\zeta_1(t_m))^2}{4} \right) = \frac{1}{2} \zeta_i(t_m) \zeta_j(t_m) - \delta_{ij} \left(1 - \frac{\varepsilon_*}{\varepsilon_0}\right) \\
 &\quad \times \left(\sum_{l=0}^{n-1} [Q(m, l+n) \zeta_1(s_{l+n}) v_1(s_{l+n}) \sigma(s_{l+n}) + Q(m, l) \zeta_1(s_l) v_1(s_l) \sigma(s_l)] \right. \\
 &\quad + \frac{1}{2n} \sum_{l=0}^{n-1} [K_3(t_m, s_{l+n}) \zeta_1(s_{l+n}) v_1(s_{l+n}) \sigma(s_{l+n}) + K_3(t_m, s_l) \zeta_1(s_l) v_1(s_l) \sigma(s_l)] \\
 &\quad \left. + \frac{\pi}{n} \sum_{l=0}^{n-1} [K_1(t_m, s_{l+n}) \frac{(\zeta_1(s_{l+n}))^2}{2} \sigma(s_{l+n}) + K_1(t_m, s_l) \frac{(\zeta_1(s_l))^2}{2} \sigma(s_l)] + \frac{(\zeta_1(t_m))^2}{4} \right) = b_{ij}(t_m).
 \end{aligned} \tag{58}$$

The above identity proves that the right-hand side vector in (50) is in the nullspace of $J - I_{2n}$, thereby concluding the proof of Proposition 5.2. \square

5.4. Final calculations yielding the terms of order 3

After evaluating $\hat{v}_{1i}^\mu, \frac{\partial \hat{v}_{1i}^\mu}{\partial v} \Big|_-$ and $\hat{v}_{2ij}^{\mu, \varepsilon}$ numerically we have to discretize the integrals in (18) and (19). Next we have to rewrite $M_{ijk}^{2,1}$ with integrals on ∂B using $\hat{v}_{2ij}^{\mu, \varepsilon}$ instead of $\frac{\partial \hat{v}_{2ij}^{\mu, \varepsilon}}{\partial v} \Big|_-$. Repeated use of the divergence theorem yields

$$M_{ijk}^{2,1} = \int_{\partial B} \hat{v}_{2ij}^{\mu, \varepsilon} v_k \, ds(\zeta).$$

Now all the integrals in the definitions of $M_{ijk}^{1,2}$ and $M_{ijk}^{2,1}$ involve smooth functions integrated along the smooth contour ∂B . Using the parametrization $(x(t), y(t)), 0 \leq t < 2\pi$ of ∂B we have to numerically evaluate integrals over $[0, 2\pi]$ of 2π periodic smooth functions. It is well known that this is best done by applying the usual trapezoidal rule.

In the case where ∂B is symmetric about its center we derive for m in $\{0, \dots, n - 1\}$

$$\zeta_k^2(t_{m+n}) v_k(t_{m+n}) \sigma(t_{m+n}) = -\zeta_k^2(t_m) v_k(t_m) \sigma(t_m)$$

and with the help of Proposition 5.2

$$\begin{aligned} \frac{\partial \hat{v}_{1i}^\mu}{\partial \mathbf{v}} \Big|_{-} (\zeta(t_{m+n})) \zeta_j(t_{m+n}) \zeta_k(t_{m+n}) \sigma(t_{m+n}) &= - \frac{\partial \hat{v}_{1i}^\mu}{\partial \mathbf{v}} \Big|_{-} (\zeta(t_m)) \zeta_j(t_m) \zeta_k(t_m) \sigma(t_m) \hat{v}_{2ij}^{\mu,\varepsilon}(\zeta(t_{m+n})) v_k(t_{m+n}) \sigma(t_{m+n}) \\ &= - \hat{v}_{2ij}^{\mu,\varepsilon}(\zeta(t_m)) v_k(t_m) \sigma(t_m). \end{aligned}$$

It is now easy to see how the terms in the sums discretizing

$$\frac{1}{2} \int_{\partial B} \frac{\partial \hat{v}_{1i}^\mu}{\partial \mathbf{v}} \Big|_{-} \zeta_j \zeta_k \, \mathbf{d}s(\zeta)$$

and

$$\int_{\partial B} \hat{v}_{2ij}^{\mu,\varepsilon} v_k \, \mathbf{d}s(\zeta)$$

cancel out two by two.

Two more integrals appear in the formulas (20) and (21). We first turn them into boundary integrals using the divergence theorem:

$$\begin{aligned} \int_B \zeta_j \, \mathbf{d}\zeta &= \int_{\partial B} \frac{1}{2} \zeta_j^2 v_j \, \mathbf{d}s(\zeta), \\ \int_B \hat{v}_{1i}^\mu \, \mathbf{d}\zeta &= \int_{\partial B} \hat{v}_{1i}^\mu \zeta_i v_i \, \mathbf{d}s(\zeta) - \frac{1}{2} \int_{\partial B} \frac{\partial \hat{v}_{1i}^\mu}{\partial \mathbf{v}} \Big|_{-} \zeta_i^2 \, \mathbf{d}s(\zeta). \end{aligned} \tag{59}$$

If ∂B is symmetric about its center the same arguments developed in the preceding paragraph apply to the right-hand side integrals in (59), they are exactly equal to 0 at the discrete level. It is now clear why the order of magnitude of the numbers in the last column of Table 1 is so small compared to that of the numbers in the center column, for all values of the number of interpolation points $2n$.

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